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Dirac operators on quantum $SU(2)$ group and quantum sphere

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Abstract

Definition of Dirac operators on the quantum group $SU_q(2)$ and the quantum sphere $S^2_{q\mu}$ are discussed. In both cases similar $SU_q(2)$-invariant form is obtained. It is connected with corresponding Laplace operators.

0. Introduction

Dirac operator undoubtedly is one of the basic notions of the A. Connes approach to noncommutative differential geometry [1]. Therefore it is natural to define Dirac operators on quantum groups and quantum homogeneous spaces as one of the most important and studied examples of quantum manifolds. As it will be shown in this paper these two problems are closely connected. For instance the same formula defines Dirac operator both on the quantum group $SU_q(2)$ and on the quantum 2-sphere $S^2_{q\mu}$ [3].

In the paper [4] the Dirac operator on the special case of quantum sphere was proposed. The leading idea of this approach was to embed quantum sphere inside quantum three dimensional Euclidean space and then to construct in this space operator commuting with the radius of quantum sphere and in the commutative limit coinciding with the standard Dirac operator on the usual two dimensional sphere.

In the proposed approach differential structures on $SU_q(2)$ and $S^2_{q\mu}$ are introduced according to the right $SU_q(2)$-coaction. That is why Dirac operators on $SU_q(2)$ and $S^2_{q\mu}$ have the similar form. The connection between [4] and the proposed approach will be studied in the next paper.

The q-Dirac operators appeared also in the study of the q-deformed Poincare groups and q-Minkowski spaces (see a review [13]). Our covariance approach is similar to the one used in [13].

Recently the notions of noncommutative geometry were used to construct noncommutative manifold started from standard sphere and the representation theory of the $SU(2)$ group [14]. The regularization parameter is connected with the highest spin. It has to be pointed out that due to the complete analogy of the su(2) algebra representation theory and the quantum $\mathrm{su}_q(2)$ one and above mentioned q-Dirac operator, the corresponding constructions can be extended to the quantum sphere case giving rise to extra parameters in the theory.

The paper is organized as follows. In Sect. 2 we give a detailed notion on the quantum group $SU_q(2)$ and the quantum sphere $S^2_{q\mu}$. Sect. 3 is devoted to the notion of the quantum enveloping algebra $\mathrm{su}_q(2)$. In Sect. 4 we give the notion of the algebra of functions on the $SU_q(2)$ quantum cotangent bundle and the analogous definition of the quantum 2-sphere cotangent bundle. And finally in Sect. 5 the $SU_q(2)$-covariant construction of Dirac operator is proposed.

1. Quantum $SU_q(2)$ group and quantum 2-sphere $S^2_{q\mu}$

The algebra of functions on the quantum group $F_q(G) = \mathrm{Fun}(SU_q(2))$ is an associative *-algebra generated by two elements a and b satisfying relations [2] ($\lambda \equiv q - \frac{1}{q}$)

$$
\begin{aligned}
&ab = qba, \qquad ab^* = qb^*a, \qquad b^*b = bb^*, \\
&a^*a - aa^* = \frac{1}{q}\lambda b^*b, \qquad aa^* + b^*b = I\,.
\end{aligned}
\tag{1.1}
$$

These relations can be written in a compact matrix form [2]

$$
\begin{aligned}
\hat{R}T_1T_2 &= T_1T_2\hat{R}, \\
\det{}_qT &= aa^* + b^*b = I\,,
\end{aligned}
\tag{1.2}
$$

where $T_1 = T \otimes I_2$, $T_2 = I_2 \otimes T$ and I_2 is the 2×2 unit matrix,

$$
T = \begin{pmatrix} a & b \\ -\frac{1}{q}b^* & a^* \end{pmatrix}
\tag{1.3}
$$

is the standard matrix of generators of $F_q(G)$ and the R-matrix

$$
\hat{R} = \begin{pmatrix} q & & & \\ & \lambda & 1 & \\ & 1 & & \\ & & & q \end{pmatrix}
\tag{1.4}
$$

satisfies the Yang-Baxter equation (in the braid group form)

$$(\hat{R}\otimes I_2)(I_2\otimes\hat{R})(\hat{R}\otimes I_2)=(I_2\otimes\hat{R})(\hat{R}\otimes I_2)(I_2\otimes\hat{R}) \tag{1.5}$$

and the Hecke condition

$$\hat{R}^2=\lambda\hat{R}+I_4\,, \tag{1.6}$$

where I_4 is the 4×4 unit matrix.

Comultiplication

$$\triangle T=T(\otimes)T \tag{1.7}$$

(i.e. $\triangle T_{ik}=\sum_j T_{ij}\otimes T_{jk}$), antipode

$$S(T)=\begin{pmatrix} a^* & -\frac{1}{q}b \\ b^* & a \end{pmatrix} \tag{1.8}$$

and counit $\varepsilon(T)=I_2$ define the structure of Hopf algebra on $F_q(G)$ [2]. We have also a useful relation

$$S(T)T=TS(T)=I_2\,. \tag{1.9}$$

The algebra of functions on quantum 2-sphere $F_q(S)=\mathrm{Fun}(S^2_{q\mu})$ is an associative *-algebra with three generators $x_+\,,x_-\,,x_3$ and two parameters q,μ, satisfying the following relations [3]

$$\begin{aligned}
x_+^*=x_-,\qquad x_3^* &= x_3, &(1.10)\\
qx_3x_+-\frac{1}{q}x_+x_3 &= \mu x_+,\\
\lambda x_3^2+\frac{1}{[2]_q}(x_+x_--x_-x_+) &= \mu x_3,\\
qx_-x_3-\frac{1}{q}x_3x_- &= \mu x_-,\\
x_3^2+\frac{1}{[2]_q}(qx_-x_++\frac{1}{q}x_+x_-) &= r^2 &(1.11)
\end{aligned}$$

where r^2 is the central element of $F_q(S)$. These relations can be written also in a compact matrix form [8]

$$M=\begin{pmatrix} \frac{1}{q}x_3 & x_- \\ x_+ & -qx_3 \end{pmatrix} \tag{1.12}$$

$$\begin{aligned}
M^\dagger &= M, &(1.13)\\
[\hat{R},(M_2\hat{R}M_2+\mu qM_2)] &= 0,\\
\frac{1}{[2]_q}\mathrm{tr}_qM^2 &= r^2 &(1.14)
\end{aligned}$$

where $[N]_q = \frac{1}{\lambda}(q^N - q^{-N})$ and the q-trace for arbitrary 2×2 matrix X is defined by $\mathrm{tr}_q X = \mathrm{tr} DX$ for $D = \mathrm{diag}(q, \frac{1}{q})$. It has an invariance property

$$\mathrm{tr}_q S(T)XT = \mathrm{tr}_q X \tag{1.15}$$

where X is an arbitrary 2×2 matrix, whose elements are commuting with elements of T.

The relations (1.11), (1.14) can be represented in the form of the reflection equation [7], [8]:

$$\begin{aligned} \hat{R}\mathbb{M}_2\hat{R}\mathbb{M}_2 &= \mathbb{M}_2\hat{R}\mathbb{M}_2\hat{R}, \\ \frac{1}{[2]q}\mathrm{tr}_q\mathbb{M}^2 &= (\mu^2 q^2 + \lambda^2 r^2) \end{aligned} \tag{1.16}$$

where $\mathbb{M} = \mu q I_2 + \lambda M$, thus omitting the condition $\mathrm{tr}_q M = 0$.

The following right coaction of the quantum group $\varphi_R : F_q(S) \longrightarrow F_q(S) \otimes F_q(G)$

$$\begin{aligned} \varphi_R(\mathbb{M}) &= S(T)\mathbb{M}T, \\ (\varphi_R(\mathbb{M}_{ij}) &= \sum \mathbb{M}_{kl} \otimes S(T)_{ik}T_{lj}) \end{aligned} \tag{1.17}$$

according to eqs. (1.2), (1.16) defines on $F_q(S)$ structure of $F_q(G)$-comodule algebra.

In the commutative case $q = 1$, $\mu = 0$ eqs. (1.12) give

$$x_1^2 + x_2^2 + x_3^2 = r^2 \tag{1.18}$$

for $x_1 = \frac{1}{2}(x_+ + x_-)$ and $x_2 = \frac{1}{2i}(x_+ - x_-)$, while for $\mu \neq 0$ one gets the Lie algebra $sl(2)$.

Invariant scalar products $\langle \cdot, \cdot \rangle_G$ on $F_q(G)$ and $\langle \cdot, \cdot \rangle_S$ on $F_q(S)$ can be defined as follows [11],[15]:

$$\begin{aligned} \langle u_G, v_G \rangle_G &= h_G(u_G^* v_G), \\ \langle u_S, v_S \rangle_S &= h_S(u_S^* v_S), \\ u_G, v_G \in F_q(G), &\qquad u_S, v_S \in F_q(S). \end{aligned} \tag{1.19}$$

The map h_G is the Haar measure on $F_q(G)$ [11], i.e. the positive linear functional $h_G : F_q(G) \to \mathbb{C}$ invariant under the coproduct

$$\begin{aligned} (id \otimes h_G)\triangle(u_G) &= h_G(u_G)I, \\ (h_G \otimes id)\triangle(u_G) &= h_G(u_G)I, \\ u_G \in F_q(G). & \end{aligned} \tag{1.20}$$

and h_S is the φ_R-invariant positive linear functional on $F_q(S)$ [15]

$$\begin{aligned} (h_S \otimes id)\varphi_R(u_S) &= h_S(u_S)I, \\ u_S \in F_q(S)\,. & \end{aligned} \tag{1.21}$$

2. Quantum universal enveloping algebra $\mathrm{su}_q(2)$

Quantum universal enveloping algebra $\mathrm{su}_q(2)$ is an associative algebra generated by four elements k, k^{-1}, e, f and relations [2],[9]

$$\begin{aligned} kk^{-1} &= I, \qquad k^{-1}k = I, \\ ek &= qke, \qquad kf = qfk, \\ k^2 - k^{-2} &= \lambda(fe - ef) \end{aligned} \tag{2.1}$$

or in the matrix form [2]

$$\begin{aligned} \hat{R}L_2^{\pm}L_1^{\pm} &= L_2^{\pm}L_1^{\pm}\hat{R}\,, \\ \hat{R}L_2^{+}L_1^{-} &= L_2^{-}L_1^{+}\hat{R}\,, \\ \det{}_q L^{\pm} &= I\,, \end{aligned} \tag{2.2}$$

where

$$L^+ = \begin{pmatrix} k^{-1} & \frac{\lambda}{\sqrt{q}}f \\ 0 & k \end{pmatrix}, \quad L^- = \begin{pmatrix} k & 0 \\ -\lambda\sqrt{q}e & k^{-1} \end{pmatrix} \tag{2.3}$$

and $\hat{R}$ is given by (1.4).

Comultiplication

$$\begin{aligned} \triangle k = k \otimes k, \qquad \triangle k^{-1} = k^{-1} \otimes k^{-1}, \\ \triangle e = e \otimes k + k^{-1} \otimes e, \qquad \triangle f = f \otimes k + k^{-1} \otimes f \end{aligned} \tag{2.4}$$

and counit map

$$\begin{aligned} \varepsilon(k) = 1, \qquad \varepsilon(k^{-1}) = 1, \\ \varepsilon(e) = 0, \qquad \varepsilon(f) = 0 \end{aligned} \tag{2.5}$$

or in the compact matrix form $\triangle L^{\pm} = L^{\pm}(\otimes)L^{\pm}$ and $\varepsilon(L^{\pm}) = I_2$ and antipode

$$S(L^+) = \begin{pmatrix} k & -\lambda\sqrt{q}f \\ 0 & k^{-1} \end{pmatrix}, \qquad S(L^-) = \begin{pmatrix} k^{-1} & 0 \\ \frac{\lambda}{\sqrt{q}}e & k \end{pmatrix} \tag{2.6}$$

define the structure of Hopf algebra on $\mathrm{su}_q(2)$. As in the $F_q(G)$ case we have relations

$$L^{\pm}S(L^{\pm}) = S(L^{\pm})L^{\pm} = I_2 \tag{2.7}$$

The Hopf algebres $F_q(G)$ and $\mathrm{su}_q(2)$ are related by the duality [2]. This means that there exists a pairing $\langle\cdot,\cdot\rangle : \mathrm{su}_q(2)\otimes F_q(G)\to\mathbb{C}$ satisfying relations:

$$\langle t_1 t_2, v\rangle = \langle t_1\otimes t_2, \triangle v\rangle. \tag{2.8a}$$

$$\langle t, v_1 v_2\rangle = \langle \triangle(t), v_1\otimes v_2\rangle. \tag{2.8b}$$

$$\langle t, S(v)\rangle = \langle S(t), v\rangle. \tag{2.8c}$$

$$\langle I, v\rangle = \varepsilon(v). \tag{2.8d}$$

$$\langle t, I\rangle = \varepsilon(t). \tag{2.8e}$$

$$t, t_1, t_2 \in U, \qquad v, v_1, v_2 \in F_q(G).$$

Using the matrix T of generators the pairing between $F_q(G)$ and $\mathrm{su}_q(2)$ can be defined by [2]:

$$\langle k, T\rangle = \begin{pmatrix} \frac{1}{\sqrt{q}} & 0 \\ 0 & \sqrt{q} \end{pmatrix}, \qquad \langle k^{-1}, T\rangle = \begin{pmatrix} \sqrt{q} & 0 \\ 0 & \frac{1}{\sqrt{q}} \end{pmatrix},$$
$$\langle e, T\rangle = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \langle f, T\rangle = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{2.9}$$

or in the compact matrix form [2]

$$\langle L_1^{\pm}, T_2\rangle = R^{\pm} \tag{2.10}$$

where

$$R^{+} = \frac{1}{\sqrt{q}}\hat{R}\mathcal{P}, \qquad R^{-} = \sqrt{q}\hat{R}^{-1}\mathcal{P} \tag{2.11}$$

and

$$\mathcal{P} = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix}$$

is a 4×4 permutation matrix in $\mathbb{C}^2\otimes\mathbb{C}^2$, so that for every 2×2 matrix $X : X_2 = \mathcal{P}X_1\mathcal{P}$.

The center of $\mathrm{su}_q(2)$ is generated by the element

$$C_q = \frac{1}{q}k^2 + qk^{-2} + \lambda^2 fe\,. \tag{2.12}$$

Inside the algebra $\mathrm{su}_q(2)$ there exists an important subalgebra generated by elements of the matrix

$$\mathbb{L} = L^{+}S(L^{-})\,. \tag{2.13}$$

The matrix $\mathbb{L}$ satisfies the reflection equation (see e.g. [5],[7])

$$\hat{R}\mathbb{L}_2\hat{R}\mathbb{L}_2 = \mathbb{L}_2\hat{R}\mathbb{L}_2\hat{R}\,. \tag{2.14}$$

The reflection equation (2.14) is invariant under the right $SU_q(2)$-coaction

$$\varphi_R(\mathbb{L}) = S(T)\mathbb{L}T\,, \tag{2.15}$$

hence the entries of $\mathbb{L}$ are generators of a right $SU_q(2)$-comodule algebra. In terms of (2.1), (2.3) it is

$$\mathbb{L} = \begin{pmatrix} k^{-2} + \frac{1}{q}\lambda^2 fe & \frac{1}{\sqrt{q}}\lambda fk \\ \frac{1}{\sqrt{q}}\lambda ke & k^2 \end{pmatrix} \tag{2.16}$$

and the central element is

$$C_q = \mathrm{tr}_q\mathbb{L}\,. \tag{2.17}$$

According to (2.15), (2.17) and (1.15) this C_q is the invariant element of this coaction.

$$\varphi_R(C_q) = C_q \otimes I\,. \tag{2.18}$$

We can also represent matrix $\mathbb{L}$ in the form

$$\mathbb{L} = \frac{1}{[2]_q}C_q I_2 + \frac{\lambda}{q}L_q \tag{2.19}$$

with the traceless matrix L_q , $tr_q L_q = 0$

$$L_q = \begin{pmatrix} \frac{1}{q}l_{q3} & l_{q-} \\ l_{q+} & -ql_{q3} \end{pmatrix} \tag{2.20}$$

and

$$\begin{aligned} l_{q+} = \sqrt{q}ke, \qquad l_{q-} &= \sqrt{q}fk, \\ l_{q3} &= \frac{1}{[2]_q}(qef - \frac{1}{q}fe) \end{aligned} \tag{2.21}$$

It follows from (2.14), (2.17) and (1.6)

$$[\hat{R}, (L_{q2}\hat{R}L_{q2} + \frac{qC_q}{[2]_q}L_{q2})] = 0 \tag{2.22}$$

or in detail

$$\begin{aligned} ql_{q3}l_{q+} - \frac{1}{q}l_{q+}l_{q3} &= \frac{1}{[2]_q}C_q l_{q+}, \\ \lambda l_{q3}^2 + \frac{1}{[2]_q}(l_{q+}l_{q-} - l_{q-}l_{q+}) &= \frac{1}{[2]_q}C_q l_{q3}, \\ ql_{q-}l_{q3} - \frac{1}{q}l_{q3}l_{q-} &= \frac{1}{[2]_q}C_q l_{q-} \end{aligned} \tag{2.23}$$

Having in mind the action of the $su_q(2)$ on the $SU_q(2)$ let us define the $SU_q(2)$-invariant Laplace operator by

$$\Delta_q = \frac{1}{[2]_q}\mathrm{tr}_q L_q^2 \,. \tag{2.24}$$

The explicit calculation gives

$$\Delta_q = l_{q3}^2 + \frac{1}{[2]_q}(q l_{q-} l_{q+} + \frac{1}{q} l_{q+} l_{q-}) \tag{2.25}$$

or, using (2.1)

$$\Delta_q = \frac{1}{\lambda^2 [2]_q^2}(C_q + [2]_q)(C_q - [2]_q)\,. \tag{2.26}$$

Due to the characteristic equation for $\mathbb{L}$ [8] we also have

$$L_q^2 + \frac{C_q}{[2]_q} L_q = I_2 \otimes \Delta_q \,. \tag{2.27}$$

As in the usual $q = 1$ case irreducible representations of $\mathrm{su}_q(2)$ are parametrized by the spin $l = 0, \frac{1}{2}, 1, ...$ The explicit formulas are [9]

$$\begin{aligned} k \mid l,m\rangle &= q^{-m} \mid l,m\rangle, \\ e \mid l,m\rangle &= \sqrt{[l-m]_q [l+m+1]_q} \mid l,m+1\rangle, \\ f \mid l,m\rangle &= \sqrt{[l-m+1]_q [l+m]_q} \mid l,m-1\rangle \end{aligned} \tag{2.28}$$

then one gets for $l_{q\pm}$ and l_{q3}

$$\begin{aligned} l_{q+} \mid l,m\rangle &= \sqrt{q^{-(2m+1)}[l-m]_q [l+m+1]_q} \mid l,m+1\rangle, \\ l_{q3} \mid l,m\rangle &= \frac{1}{[2]_q} q^{-m}(q^{l+1}[l+m]_q - q^{-(l+1)}[l-m]_q) \mid l,m\rangle, \\ l_{q-} \mid l,m\rangle &= \sqrt{q^{-(2m-1)}[l-m+1]_q [l+m]_q} \mid l,m-1\rangle \end{aligned} \tag{2.29}$$

as well as for C_q and Δ_q

$$\begin{aligned} C_q \mid l,m\rangle &= (q^{2l+1} + q^{-(2l+1)}) \mid l,m\rangle, \\ \Delta_q \mid l,m\rangle &= [l]_{q^2}[l+1]_{q^2} \mid l,m\rangle \end{aligned} \tag{2.30}$$

where each multiplet $\mid l,m\rangle$, $m = -l, -l+1, ..., l$ forms the basis of the corresponding irreducible representation space V_l.

3. $SU_q(2)$ and $S^2_{q\mu}$ as noncommutative manifolds

The right $SU(2)_q$-comodule structures on $F_q(G)$ and $F_q(S)$ (1.7), (1.17) define the corresponding left $\mathrm{su}_q(2)$-module structures on them [2],[16]. It means that for every $\psi \in \mathrm{su}_q(2)$

$$\hat{\psi}^G(u) = (id \otimes \psi)\triangle(u), \qquad u \in F_q(G) \tag{3.1}$$

$$\hat{\psi}^S(v) = (id \otimes \psi)\varphi_R(v), \qquad v \in F_q(S) \tag{3.2}$$

and the maps $\hat{\psi}^G$ and $\hat{\psi}^S$ act as linear operators on elements of $F_q(G)$ and $F_q(S)$. Since the mappings $\psi \to \hat{\psi}^{G,S}$ are homomorphisms [5],[6] all the relations (2.1), (2.2), (2.14) are valid for $\hat{k}^{G,S}$, $\hat{k}^{-1G,S}$, $\hat{e}^{G,S}$, $\hat{f}^{G,S}$, $\hat{L}^{\pm G,S}$ and $\hat{L}^{G,S}$.

The elements of $F_q(G)$ and $F_q(S)$ can be also considered as left multiplication operators on these spaces. Then as it was shown in [5], [6] the action on the quantum group of the dual quantum algebra can be written as

$$q\hat{\mathbb{L}}_1^G T_2 = T_2 \hat{R} \hat{\mathbb{L}}_2^G \hat{R}\,. \tag{3.3}$$

Using the same technique as in one can derive the corresponding relation for the quantum sphere with $\hat{\mathbb{L}}^S$ and M [7], [8]

$$\hat{R}\hat{\mathbb{L}}_2^S \hat{R} M_2 = M_2 \hat{R}\hat{\mathbb{L}}_2^S \hat{R}\,. \tag{3.4}$$

It is easy to see from (1.12), (2.23) and (3.4) that operator

$$\mathcal{K} = \lambda^2 \mathrm{tr}_q M\mathbb{L} + \mu(1_S - \frac{\hat{C}_q}{[2]_q}) \tag{3.5}$$

commute both with M and $\mathbb{L}$ and $\mathcal{K}(1_S) = 0$. So it is a zero operator.

The joint algebra with the entries of $\hat{\mathbb{L}}^G$ and T as the generators and relations (1.2), (2.14), (3.3) defines the algebra of functions on quantum cotangent bundle of $SU_q(2)$ [5], [6]. We shall denote it by $F_q(T^*G)$. By the same arguments the joint algebra with generators as the entries of $\hat{\mathbb{L}}^S$ and M , relations (1.14), (2.14),(3.4) and the additional relation:

$$\mathcal{K} = 0 \tag{3.6}$$

is called the algebra of functions on the quantum cotangent bundle of $S^2_{q\mu}$ and is denoted by $F_q(T^*S)$.

The right coaction

$$\varphi_R(T) = \triangle T, \qquad \varphi_R(\hat{\mathbb{L}}^{G,S}) = S(T)\hat{\mathbb{L}}^{G,S} T, \qquad \varphi_R(M) = S(T)MT \tag{3.7}$$

define on $F_q(T^*G)$ and $F_q(T^*S)$ structures of right $SU_q(2)$-comodule algebras.

Scalar products $\langle\cdot,\cdot\rangle_G$ on $F_q(G)$ and $\langle\cdot,\cdot\rangle_S$ on $F_q(S)$ defines *-conjugation on these algebras. On T and M it coincides with the initial definition

$$T^\dagger = S(T)\,, \qquad M^\dagger = M \tag{3.8}$$

and as it will be proved in Appendix the corresponding formulas for $\hat{L}^{\pm}$ are

$$(\hat{L}^{\pm})^{\dagger} = S(\hat{L}^{\mp}) \tag{3.9}$$

(where indices "G,S" are omitted). From (3.7) we have

$$\hat{k}^* = \hat{k}, \qquad \hat{e}^* = \hat{f} \tag{3.10}$$

From (2.13), (2.17) and (2.19) we also have

$$\hat{\mathbb{L}}^{\dagger} = \hat{\mathbb{L}}, \qquad \hat{L}_q^{\dagger} = \hat{L}_q, \qquad \hat{C}_q^* = \hat{C}_q. \tag{3.11}$$

Now describe the $q \to 1$ limit. Let

$$\hat{L}^{G,S} = \begin{pmatrix} \hat{l}_3^{G,S} & \hat{l}_-^{G,S} \\ \hat{l}_+^{G,S} & -\hat{l}_3^{G,S} \end{pmatrix}$$

and $\hat{C}^{G,S}$ defines the $q \to 1$ limits of $\hat{L}_q^{G,S}$ and $\hat{C}_q^{G,S}$. Then the direct calculation using (3.1), (3.2), (2.4), (2.8b) and (2.9) gives

$$\hat{C}^{G,S} = 2 \tag{3.12}$$

$$\begin{aligned}
\hat{l}_+^G &= a\frac{\partial}{\partial b} - b^*\frac{\partial}{\partial a^*}, \\
\hat{l}_3^G &= \frac{1}{2}(a\frac{\partial}{\partial a} + b^*\frac{\partial}{\partial b^*} - b\frac{\partial}{\partial b} - a^*\frac{\partial}{\partial a^*}), \\
\hat{l}_-^G &= b\frac{\partial}{\partial a} - a^*\frac{\partial}{\partial b^*}
\end{aligned} \tag{3.13}$$

and

$$\hat{l}_k^S = \frac{1}{i}\varepsilon_{kmn}x_m\frac{\partial}{\partial x_n} \tag{3.14}$$

where ε_{kmn} is the Levi-Civita tensor with $\varepsilon_{123} = 1$ and

$$\hat{l}_{\pm}^{G,S} = \hat{l}_1^{G,S} \pm i\hat{l}_2^{G,S}$$

Operators $\hat{l}_k^{G,S}$ satisfy the su(2) Lie algebra commutation relations

$$[\hat{l}_k^{G,S}, \hat{l}_m^{G,S}] = i\varepsilon_{kmn}\hat{l}_n^{G,S} \tag{3.15}$$

and the $q \to 1$ limit of $\hat{\Delta}_q^{G,S}$ coincides with the corresponding Casimir operator

$$\hat{\Delta} = \hat{l}_1^2 + \hat{l}_2^2 + \hat{l}_3^2 \tag{3.16}$$

(where we again omitted indices "G,S"), and the equation (3.6) gives

$$\sum_{k=1}^{3} x_k\hat{l}_k = \frac{1}{i}\sum_{i=1}^{3}\varepsilon_{kmn}x_k x_m\frac{\partial}{\partial x_n} = 0 \tag{3.17}$$

4. Dirac operators on $SU_q(2)$ and $S^2_{q\mu}$

Since all the formulas of this paragraph are similar in both cases of $SU_q(2)$ and $S^2_{q\mu}$ we shall omit indices "G" and "S".

First let us notice that according to (2.27) we can put the $\hat{L}_q$ from (2.20) as the definition of Dirac operator on $F_q(G)$ (or $F_q(S)$). We shall denote it by D_q.

$$D_q = \begin{pmatrix} \frac{1}{q} l_{q3} & l_{q-} \\ l_{q+} & -q l_{q3} \end{pmatrix} \tag{4.1}$$

The characteristic equation (2.27) gives

$$D_q^2 + \frac{\hat{C}_q}{[2]_q} D_q = I_2 \otimes \Delta_q \tag{4.2}$$

Due to (3.11) $D_q^* = D_q$ and in the case $q = 1$ the corresponding operator D has the form [4]

$$D = \sum_{i=1}^{3} \sigma_k \otimes \hat{l}_k \tag{4.3}$$

where σ_k are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and operators $\hat{l}_k$ are given by (3.12), (3.13).

The eq. (4.1) gives

$$D^2 + D = I_2 \otimes \Delta \tag{4.4}$$

and according to [10] D can be interpreted as Dirac operator on $SU(2)$ or S^2.

In order to define corresponding to D_q spinor states we shall study in detail the structure of the operator D_q.

Let us consider two representations of su$_q(2)$: the fundamental irrep π_2 and the regular one π_{reg}

$$\pi_2(k) = \begin{pmatrix} \frac{1}{\sqrt{q}} & 0 \\ 0 & \sqrt{q} \end{pmatrix}, \qquad \pi_2(k^{-1}) = \begin{pmatrix} \sqrt{q} & 0 \\ 0 & \frac{1}{\sqrt{q}} \end{pmatrix},$$
$$\pi_2(e) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \pi_2(f) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{4.5}$$

and

$$\pi_{reg}(u) = \hat{u} \tag{4.6}$$

for every $u \in \mathrm{su}_q(2)$. We need also their tensor product $\pi_{tot} = \pi_2 \otimes \pi_{reg}$ defined by

$$\pi_{tot}(u) = (\pi_2 \otimes \pi_{reg})(\triangle u) \tag{4.7}$$

for $u \in \mathrm{su}_q(2)$. Let $K = \pi_{tot}(k)$, $K^{-1} = \pi_{tot}(k^{-1})$, $E = \pi_{tot}(e)$ and $F = \pi_{tot}(f)$, then the straightforward calculation gives

$$\begin{aligned}
K &= \begin{pmatrix} \frac{1}{\sqrt{q}} & 0 \\ 0 & \sqrt{q} \end{pmatrix} \otimes \hat{k}, \qquad K^{-1} = \begin{pmatrix} \sqrt{q} & 0 \\ 0 & \frac{1}{\sqrt{q}} \end{pmatrix} \otimes \hat{k}^{-1}, \\
E &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \hat{k} + \begin{pmatrix} \sqrt{q} & 0 \\ 0 & \frac{1}{\sqrt{q}} \end{pmatrix} \otimes \hat{e}\,, \\
F &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \hat{k} + \begin{pmatrix} \sqrt{q} & 0 \\ 0 & \frac{1}{\sqrt{q}} \end{pmatrix} \otimes \hat{f}\,.
\end{aligned} \tag{4.8}$$

Operators K, K^{-1}, E and F satisfies all relations (2.1) and the corresponding central element

$$C_q^{tot} = \frac{1}{q} K^2 + qK^{-2} + \lambda^2 FE \tag{4.9}$$

can be expressed in terms of $\hat{C}_q$ and D_q

$$C_q^{tot} = \frac{[2]_{q^2}}{[2]_q} I_2 \otimes \hat{C}_q + \lambda^2 D_q. \tag{4.10}$$

Thus we see that D_q due to centrality of $\hat{C}_q$ and C_q^{tot} also commutes with K, K^{-1}, E and F, demonstrating invariance property. So we may consider the algebra $\mathcal{A}_q$ with generators K, K^{-1}, E, F and D_q. According to (4.1) and (2.26)

$$D_q^2 + \frac{\hat{C}_q}{[2]_q} D_q = \frac{1}{\lambda^2 [2]_q^2} I_2 \otimes (\hat{C}_q + [2]_q)(\hat{C}_q - [2]_q) \tag{4.11}$$

where the operator $I_2 \otimes \hat{C}_q$ also lies in $\mathcal{A}_q$ according to (4.8) and (4.9). Introducing the operators

$$\begin{aligned}
\mathcal{L}_+ &= \pi_{tot}(l_+) = \sqrt{q} KE, \\
\mathcal{L}_3 &= \pi_{tot}(l_3) = \frac{1}{[2]_q}(qEF - \frac{1}{q} FE), \\
\mathcal{L}_- &= \pi_{tot}(l_-) = \sqrt{q} FK.
\end{aligned} \tag{4.12}$$

and using (4.8), one gets

$$\mathcal{L}_+ = \frac{1}{[2]_q} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes (\hat{C}_q - \lambda [2]_q \hat{l}_3) + I_2 \otimes \hat{l}_{q+},$$

$$\mathcal{L}_3 = \frac{1}{[2]_q^2}\begin{pmatrix} q & 0 \\ 0 & -\frac{1}{q}\end{pmatrix} \otimes \hat{C}_q + \frac{2}{[2]_q} I_2 \otimes \hat{l}_3 + \frac{\lambda}{[2]_q}\begin{pmatrix} 0 & \hat{l}_- \\ \hat{l}_+ & 0\end{pmatrix},$$

$$\mathcal{L}_- = \frac{1}{[2]_q}\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix} \otimes (\hat{C}_q - \lambda[2]_q \hat{l}_3) + I_2 \otimes \hat{l}_- \tag{4.13}$$

We may consider now operators $\mathcal{L}_+$, $\mathcal{L}_3$, $\mathcal{L}_-$, $\hat{C}_q$ and D_q as generators of another $SU_q(2)$-covariant algebra $\mathcal{B}_q$. Introducing the matrix generator

$$\mathbb{L}^{tot} = \frac{1}{[2]_q} I_2 \otimes C_q^{tot} + \frac{\lambda}{q} L^{tot} \tag{4.14}$$

where

$$L^{tot} = \begin{pmatrix} \frac{1}{q}\mathcal{L}_3 & \mathcal{L}_- \\ \mathcal{L}_+ & -q\mathcal{L}_3 \end{pmatrix} \tag{4.15}$$

we can write commutation relations in the $\mathcal{B}_q$ as

$$\hat{R}^{tot}\mathbb{L}_2^{tot}\hat{R}^{tot}\mathbb{L}_2^{tot} = \mathbb{L}_2^{tot}\hat{R}^{tot}\mathbb{L}_2^{tot}\hat{R}^{tot} \tag{4.16}$$

where $\hat{R}^{tot} = \hat{R} \otimes I_2$. The right $SU_q(2)$-coaction on $\mathcal{B}_q$ is given by (2.15) and

$$\varphi_R(D_q) = D_q \otimes I, \qquad \varphi_R(\mathbb{L}^{tot}) = S(T)\mathbb{L}^{tot}T. \tag{4.17}$$

The spectrum of D_q can be easily obtained from (4.1) and (2.30). The straightforward calculation gives two series of eigenvalues

$$\lambda_l^+ = [l]_{q^2}\,, \qquad \lambda_l^- = -[l+1]_{q^2}\,. \tag{4.18}$$

Corresponding to $\lambda_l^\pm$ eigenfunctions can be obtained by decomposition of the tensor product $V_{\frac{1}{2}} \otimes V_l$ into the direct sum $V_{l+\frac{1}{2}} \oplus V_{l-\frac{1}{2}}$

$$\mid l \pm \frac{1}{2}, m\rangle = \sum_{m_1,m_2} \begin{bmatrix} \frac{1}{2} & l & l\pm\frac{1}{2} \\ m_1 & m_2 & m \end{bmatrix}_q \mid \frac{1}{2}, m_1\rangle \otimes \mid l, m_2\rangle \tag{4.19}$$

where $\begin{bmatrix} \frac{1}{2} & l & l\pm\frac{1}{2} \\ m_1 & m_2 & m \end{bmatrix}_q$ are the quantum Clebsch-Gordan coefficients [12] Using decomposition of $F_q(G)$ and $F_q(S)$ on irreducible subspaces under the $\mathrm{su}_q(2)$ action [15],[16] we can express vectors $\mid l, m_2\rangle$ of (4.19) in terms of q-special functions.

According to (2.30) spaces $V_{l\pm\frac{1}{2}}$ are eigenspaces of C_q^{tot} with eigenvalues $q^{2(l\pm\frac{1}{2})+1} + q^{-(2(l\pm\frac{1}{2})+1)}$. Since in our case $V_{l\pm\frac{1}{2}}$ are imbedded into $V_{\frac{1}{2}} \otimes V_l$ they also are eigenspaces of $I_2 \otimes \hat{C}_q$ with eigenvalue $q^{2l+1} + q^{-(2l+1)}$. So the eq. (4.10) gives

$$D_q \mid l \pm \frac{1}{2}, m\rangle = \lambda_l^\pm \mid l \pm \frac{1}{2}, m\rangle. \tag{4.20}$$

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Appendix A.

Let us prove the general relation

$$ {}^{*}\circ \hat{L}^{\pm} \circ^{*} = (\hat{L}^{\mp})^{t} \tag{A.1}$$

where "$*$" is the *-conjugation in $F_q(G)$ or $F_q(S)$ and "t" means the usual 2×2-matrix transposition.

In the $F_q(G)$ case (A.1) means the following

$$ (\hat{L}_1^{\pm G}(T_2^c))^c = (\hat{L}_1^{\mp G})^t(T_2) \tag{A.2}$$

where $T^c = ST^t$. So from (2.10) we have $\hat{L}_1^{\pm G}(T_2^c) = S\hat{L}_1^{\pm G}(T_2^t)$, and direct calculation using the technique of [6] gives

$$ \hat{L}_1^{\pm G}(T_2^c) = T_2^c((R^{\pm})^{-1})^{t_2} \tag{A.3}$$

where "t_2" is the transposition in the second space. For the right hand side of (A.2) we have

$$ (\hat{L}_1^{\pm G})^t(T_2) = T_2(R^{\mp})^{t_1} \tag{A.4}$$

where "t_1" means transposition in the first space. Eqs. (A.2), (A.3) and (A.4) give

$$ (R^{\mp})^{t_1} = ((R^{\pm})^{-1})^{t_2} \tag{A.5}$$

or

$$ (R^{\pm})^{-1} = (R^{\mp})^{t} \tag{A.6}$$

and from (2.11) we get a very simple relation

$$ \hat{R}^t = \hat{R} \tag{A.7}$$

which follows immediately from the explicit expression (1.4).

In the $F_q(S)$ case eq. (A.1) means

$$ (\hat{L}^{\pm S}(M_2^t))^c = (\hat{L}^{\mp S})^t(M_2) \tag{A.8}$$

Following [6] we can easily obtain

$$ \hat{L}_1^{\pm S}(M_2) = (R^{\pm})^{-1}M_2R^{\pm} \tag{A.9}$$

So from (A.8) and (A.9) it follows

$$ (R^{\pm})^{t_2}M_2((R^{\pm})^{-1})^{t_2} = ((R^{\mp})^{-1})^{t_1}M_2(R^{\mp})^{t_1} \tag{A.10}$$

which again leads to (A.6).

In terms of the generators eq. (A.1) means

$$ {}^* \circ \hat{k} \circ^* = \hat{k}^{-1}, \qquad {}^* \circ \hat{e} \circ^* = -\frac{1}{q}\hat{f} \tag{A.11} $$

Let us prove now the formula (3.10). Consider the following chains of relations using (A.1), (2.1), (2.8a), (3.1) and (3.2)

$$ \begin{gathered} \langle x, \hat{k}(y) \rangle = h(x^* \hat{k}(y)) = h \circ \hat{k}(\hat{k}^{-1}(x^*)y) = \\ = h \circ (id \otimes k)\triangle(\hat{k}^{-1}(x^*)y) = k(h \otimes id)\triangle(\hat{k}^{-1}(x^*)y) = \\ = k(I)h(\hat{k}^{-1}(x^*)y) = \langle (\hat{k}^{-1}(x^*))^* y \rangle = \langle \hat{k}(x), y \rangle \end{gathered} \tag{A.12} $$

and

$$ \begin{gathered} \langle x, \hat{e}(y) \rangle = h(x^* \hat{e}(y)) = h \circ \hat{e}(\hat{k}(x^*)y) - h(\hat{e}\hat{k}(x^*)\hat{k}(y)) = \\ = e \circ (h \otimes id)\triangle(\hat{k}(x^*)y) - qh \circ \hat{k}(\hat{e}(x^*)y) = \\ = e(I)h(\hat{k}(x^*)y) - qh \circ (id \otimes k)\triangle(\hat{e}(x^*)y) = \\ = -qk(h \otimes id)\triangle(\hat{e}(x^*)y) = -qh(\hat{e}(x^*)y) = \\ -q\langle (\hat{e}(x^*))^*, y \rangle = \langle \hat{f}(x), y \rangle. \end{gathered} \tag{A.13} $$

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